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THE PYRAMID COLLECTION

In this article I would like to present readers with some very useful activities, ideas and problems. I have used these at various times over the years and I think that, together, they make a very interesting collection. These pieces have come together from different sources, but like a jigsaw, they complete the picture of my understanding of the numbers inside the building of triangles and pyramids.

In NSW, with the new K–10 Syllabus now being implemented, most of this material would be best placed in the Number strand. The activities on the pyramids, cairns, apex card trick and Pascal's triangle all come under the large umbrella of what is called Additional Content in the new syllabus. This is certainly 'material in order to broaden and deepen students' knowledge, skills and understanding, to meet students' interests or to stimulate student interest in other areas of mathematics' (Board of Studies, NSW, 2002, p. 14).

Senior students, doing their Preliminary and HSC Courses in General Mathematics (non-calculus course) (Board of Studies, NSW, 1999) could also benefit from the use of the material. The activities on ziggurats and pyramidal numbers also fit in nicely with this syllabus in the components of Measurement: M2 (applications of area and volume) and Algebraic Modelling: AM4 (modelling linear and non-linear relationships).

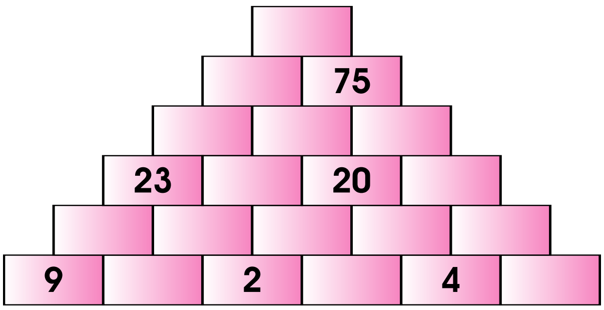
Students preparing for the Mathematics (Calculus course) and Extension One (Advanced Calculus course) will benefit from investigating sigma notation for each of the layers in the solids in ziggurats as part of the Sequences and Series topic. The formulae also provide some visual examples of 'Mathematical Induction' in the Extension One course.

Activity one: Problem solving pyramid

There are many pyramid puzzles. The two shown here are adapted from an idea in the British magazine, *Figure It Out* (Nexus Media Ltd, 17 September 1998, pp. 9, 30, 60). This magazine and other logic puzzle publications are a great source of stimulating materials for you and your students.

I have made up two examples similar to the actual ones in the magazine and your task is to fill in the missing numbers in the grid using the numbers already in the grid. The number in each brick is the 'added total' of both the bricks that it is sitting on.

Pyramid one



Give the pyramid a try. As a teacher you often need to put yourself in the same position as your students in order to be able to help them and interest them in problem solving. I hope that the number of different methods that can be used will surprise you. Do not forget to discuss all the strategies that were used by you and your students on this problem. You can quickly make up similar questions and use the same idea again.

Discussion

a) Did you 'guess and check' with numbers in the bottom row with the two smaller triangles and the numbers 23 and 20 at their apex?

		23								20		
	?		?						?		?	
9		?		2				2		?		4

b) Did you use algebra in the same triangles? Say letting the number in the bottom row, second from the left be x and then use $(x + 9) + (x + 2) = 23$?

c) Did you analyse the whole triangle giving each number in the bottom row a letter? For example, a, b, c, d, e, f. The apex number at the top of the pyramid is $a + 5b + 10c + 10d + 5e + f$.

d) Did you need to fill in the whole pyramid just to find 152 in the apex brick of the largest triangle?

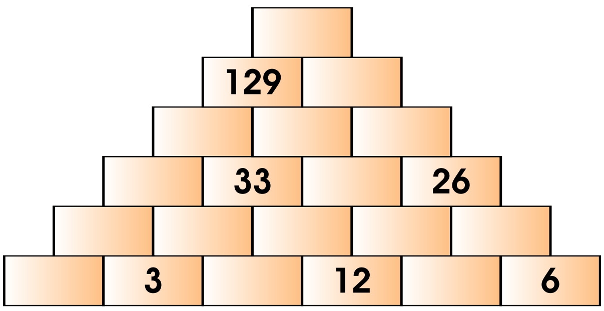
e) What different tactics did you need to use to find the numbers in the rest of the bricks?

Solution

152
77 **75**
40 37 38
23 17 **20** 18
15 8 9 11 7
9 6 **2** 7 **4** 3

Here is another pyramid to solve.

Pyramid two



Solution

262
129 133
59 70 63
26 **33** 37 **26**
14 12 21 16 10
11 **3** 9 **12** 4 **6**

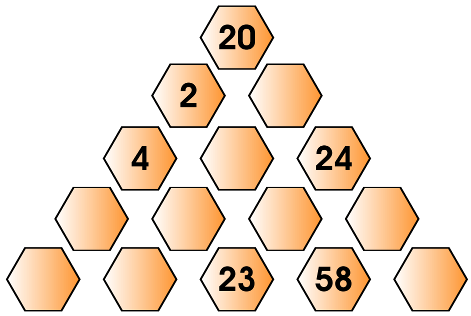
Activity two: Cairns

A similar activity is to work from the top down. A few years ago these Complete the Cairn puzzles were in the Brain Games section of *The Australian Magazine*, inside *The Australian* newspaper.

> Each of the stones in the upper rows of the cairn sits upon two lower stones. The number in each of the upper stones represents the difference between the numbers in the two stones on which it rests. What are the five two digit numbers in the bottom row of stones? (Each of the digits 0–9 is used, once only, in this row.)
> (Moodie-Bloom, 1995, p.70)

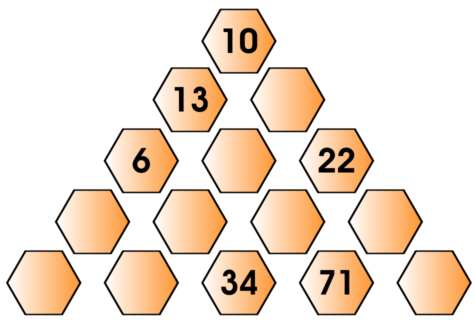
The picture of the cairn shows how similar puzzles can be drawn up quickly on cardboard, overhead, blackboard or whiteboard.

Cairn one



See what tactics you and your students use to complete the cairn.

Cairn two



Solutions

Cairn one

20

2 22

4 2 **24**

41 37 35 11

19 60 **23** **58** 47

Cairn two

10

13 3

6 19 **22**

62 56 37 15

28 90 **34** **71** 56

Hope you do the second one more quickly than I did — I kept making some wrong decisions!

Activity three

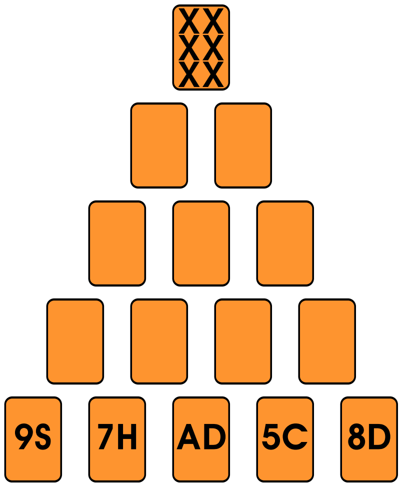
a) The apex card trick

This is another classic puzzle that unlocks the key to Pascal's Triangle. Martin Gardner has been an absolute inspiration to me over the years. If you have never read or used any of his books please start by chasing up one of his many books. From Chapter 15 in his book, *Mathematical Carnival* (Gardner, 1975) comes my favourite application of Pascal's Triangle, the Apex Card Trick, although this is also available elsewhere.

As the presenter of this effect, you lay out five cards, face up. From these five cards, predict the apex number and lie it face down before you add the numbers, in the same fashion as Pyramid Problem Solving. Do not let the class know what the predicted number is at this stage. Pick cards from the pack, and lay them out face up to build the triangle up to your selected face down card by adding two numbers to give a single digit answer.

For example, if the first five cards are a 9, 7, Ace, 5 and 8 then the predicted card at the apex that you put face down will be an eight. Where the sum of the two cards is a two digit number, add the digits again.

Example: 9 + 7 = 16, but we do not have a 16 in cards so we add again. We get 1 + 6 = 7, as the first card in the second row.



Solution

Write down the first five lines of Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

The numbers in Pascal's triangle tell you the number of paths to the apex for each of the five numbers in the bottom row. To determine the apex number, multiply the value of the first card by one, add 4 times the second card plus 6 times the third card plus 4 times the fifth card plus 1 times the sixth card. Do this calculation, but keep adding the digits in your answer as a running total. Keep going until you get a single digit answer. This method is called 'casting out nines'. You can even cast out nines as you go each time you get a sum of nine or bigger.

In our example,

$1 \times 9 + 4 \times 7 + 6 \times 1 + 4 \times 5 + 1 \times 8$
$= 9 + 28 + 6 + 20 + 8$
$= 71$ $(7 + 1)$
$= 8$

OR

1×9 $(= 0) + 4 \times 7$ $(28, 2 + 8 = 10, 1 + 0 = 1)$
$+ 6 \times 1$ $(= 6) + 4 \times 5$ $(20, 2 + 0 = 2)$
$+ 1 \times 8$ $(= 8)$ $(0 + 1 + 6 + 2 + 8)$
$= 17$
$= 8$

In discussions with your students about doing this mathematical trick you can reinforce ideas about order of operations and mental computation skill, because you need to get the answer correct in your head!

b) A prediction using ten numbers in the base

This can be even more impressive than with the cards. You can ask a volunteer from your class for ten single digit numbers and predict the apex number, writing it up on the board. Here is your starting line of ten numbers:

8, 1, 3, 5, 6, 3, 2, 1, 7, 9.

Your prediction for the apex number is…?

Solution

Using ten numbers in the base

(2)
4 7
1 3 4
8 2 1 3
9 8 3 7 5
2 7 1 2 5 9
7 4 3 7 4 1 8
4 3 1 2 5 8 2 6
9 4 8 2 9 5 3 8 7
8 1 3 5 6 3 2 1 7 9

The tenth line of Pascal's Triangle is:

1, 9, 36, **84**, 126, **84**, 36, 9, **1**

There are only four numbers in the base that do not reduce to zero by the 'casting out nines' process. They are the first (8), fourth (5), seventh (2) and tenth (9) numbers. Note that the number 84 has a remainder of 3 when divided by the number 9, or casting out nines gives 8 + 4 = 12, 1 + 2 = 3.

For our example all you need to do to make your apex number prediction is calculate

$1 \times 8 + 3 \times 5 + 3 \times 2 + 1 \times 9$
$= 8 + 15 + 6 + 9$
$= 38$ (3 + 8 = 11; 1 + 1 = 2)

Therefore number 2 is our apex number.

After working with the apex card trick and the ten number version, the 'problem solving pyramid' and 'cairns', are also more easily understood. When you start with this new set of ten numbers (8, 6, 4, 3, 1, 2, 5, 8, 9, 4) your answer for the apex number should be 9.

($1 \times 8 + 3 \times 3 + 3 \times 5 + 1 \times 4$
$= 8 + 9 + 15 + 4$
$= 8 + 6 + 4$
$= 18$
$= 9$ (Do not cast out at the end!)

Extension activities

Investigate Pascal's Triangle further to see how you could find the number at the apex for various sized triangles that have 4, 5, 6, 7, 8 or 9 numbers in the bottom row.

Activity four: Ziggurats

The Babylonians built their temples on the top of stepped pyramids called *ziggurats*. In examples from *Mathematics Cubed: Investigation with Interlocking Cubes* (Bramald, Thompson & Straker, 1997, p. 45) you are asked to build some models and develop the correct formulae to go with each of them.

a) Ziggurat 1

In the bottom layer there are 5 cubes on each side, then 4, then 3, then 2 and then 1 on top.

b) Ziggurat 2

In the bottom layer there are 7 cubes on each side, then 5, then 3 and then 1 on top.

Find formulae for the number of cubes in each layer (T_n) and the total number of cubes in each Ziggurat (S_n). You could see first if your students can work out the numbers of blocks needed to build these models 4, 5 or even ten high.

Solutions

Ziggurat 1

$$T_n = n^2$$

$$S_n = \sum_{1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

Ziggurat 2

$$Tn = (2n-1)^2$$

$$S_n = \sum_{1}^{n} (2r-1)^2 = \frac{n}{3}(2n+1)(2n-1)$$

The reader is left to use the formula, or some other method to work out the numbers needed to build the 4, 5 and 10 high Ziggurats.

With the formulae from the ziggurats and pyramids activities, the teacher is armed with a powerful visual model to demonstrate mathematics from the Sequences and Series topic.

T_n can indicate the number of blocks in each layer while S_n will be the total number of blocks in each structure of n layers. These formulae can also be extended to develop the method of mathematical induction with a visual representation of T_n and S_n.

Activity five: Pyramidal numbers

In an old book called *Math Miracles* (Wallace Lee, 1976, p. 20), 'zikkarats' are also mentioned under the heading of 'Some Odd Formulas'. Here are two more examples.

a) A square-based pyramid

A hypothetical pyramid has a bottom layer of stones laid in a perfect square up to one stone in the top layer. If there are 432 stones in each side of the bottom layer how many stones are there altogether?

b) A triangular-based pyramid (tetrahedron)

Imagine a pyramid stack of cannon balls. This may be modelled with tennis balls or similar. To make the 'tennis ball pyramids' make a frame similar to those found on a snooker or pool table to hold the solid together. Find the formula:

i) to give the number of balls in each layer

ii) to give the total number of balls when there are n layers (or n balls on each edge of the pyramid.)

Solutions

a) Square-based pyramid

Using

$$\frac{n(n+1)(2n+1)}{6}$$

gives us 26 967 240 stones!

b) Triangular-based pyramid

$$T_n = \frac{n(n+1)}{2} \quad \text{(triangular numbers)}$$

$$Sn = \frac{n(n+1)(n+2)}{6}$$

These puzzles have all created a lot of interest among students in all my classes, from Years 7–12.

Drawings on large sheets of cardboard were used for group work and A4 worksheets seemed to work best for individual work. The hands-on approach of actually building ziggurats with cubes was also appealing to groups of students.

Use www.askjeeves.com or similar to find out information such as:

- What is a cairn?
- How were the pyramids built?
- How high and how many stones were used in the Pyramids of Giza?

Many of the puzzles presented here are also now on the Internet. I hope that readers find some useful material in these triangles and pyramids and that it leads you to dig up some more motivational material for your students.

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References

Board of Studies, NSW (1999). *Stage 6 General Mathematics Syllabus for Preliminary and HSC Courses*. Sydney: Author.

Board of Studies, NSW (2002). *Mathematics Syllabus: Years 7–10*. Sydney: Author.

Bramald, R., Thompson, I. & Straker, N. (1997). *Mathematics Cubed: Investigations with Interlocking Cubes*. Colchester: Claire Publications:

Nexus Media Ltd (17 September 1998). *Figure it out, No. 6*.

Gardner, M. (1975). *Mathematical Carnival*. London: Penguin Books.

Moodie-Bloom, T. (1995). Brain games: The cairns puzzles. *The Australian Magazine*, 70.

Wallace, L. W. (1976). *Math Miracles*. Calgary: Mickey Hades International.

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